

# Backward SPDEs with non-local in time and space boundary conditions

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Submitted: November 7, 2012. Revised: November 20, 2012

## Abstract

We study linear backward stochastic partial differential equations of parabolic type with special boundary condition that connect the terminal value of the solution with a functional over the entire past solution. Uniqueness, solvability and regularity results for the solutions are obtained.

*AMS 1991 subject classification:* Primary 60J55, 60J60, 60H10. Secondary 34F05, 34G10.

*Key words and phrases:* backward SPDEs, periodic conditions, mixed in time conditions.

## 1 Introduction

Partial differential equations and stochastic partial differential equations (SPDEs) have fundamental significance for natural sciences, and various boundary value problems for them were widely studied. Usually, well-posedness of a boundary value depends on the choice of the boundary value conditions.

Boundary value problems for SPDEs are well studied in the existing literature for the case of forward parabolic Ito equations with the Cauchy condition at initial time (see, e.g., Alós et al (1999), Bally *et al* (1994), Da Prato and Tubaro (1996), Gyöngy (1998), Krylov (1999), Maslowski (1995), Pardoux (1993), Rozovskii (1990), Walsh (1986), Zhou (1992), and the bibliography there). Many results have been also obtained for the backward parabolic Ito equations with Cauchy condition at terminal time, as well as for pairs of forward and backward equations with separate Cauchy conditions at initial time and the terminal time respectively; see, e.g., Yong and Zhou (1999), and the author's papers (1992), (2005), (2011), (2012). Note that a

backward SPDE cannot be transformed into a forward equation by a simple time change, unlike as for the case of deterministic equations. Usually, a backward SPDE is solvable in the sense that there exists a diffusion term being considered as a part of the solution that helps to ensure that the solution is adapted to the driving Brownian motions.

There are also results for SPDEs with boundary conditions connecting the solution at different times, for instance, at initial time and at terminal time. This category includes stationary type solutions for forward SPDEs (see, e.g., Caraballo *et al* (2004), Chojnowska-Michalik (19987), Chojnowska-Michalik and Goldys (1995), Duan *et al* (2003), Mattingly (1999) Mohammed *et al* (2008), Sinai (1996), and the references here). There are also results for periodic solutions of SPDEs (Chojnowska-Michalik (1990), Feng and Zhao (2012), Klünger (2001)). As was mentioned in Feng and Zhao (2012), it is difficult to expect that, in general, a SPDE has a periodic in time solution  $u(\cdot, t)|_{t \in [0, T]}$  in a usual sense of exact equality  $u(\cdot, t) = u(\cdot, T)$  that holds almost surely given that  $u(\cdot, t)$  is adapted to some Brownian motion. The periodicness of the solutions of stochastic equations was usually considered in the sense of the distributions. In Feng and Zhao (2012), the periodicness was established in a stronger sense as a "random periodic solution (see Definition 1.1 from Feng and Zhao (2012)). Dokuchaev (2012) considered backward SPDEs with quite general non-local and time and space boundary conditions. These conditions cover a setting where periodicness condition hold almost surely, as well as more general conditions  $\kappa u(\cdot, 0) = u(\cdot, T) + \xi$  a.e., where  $\kappa \in [-1, 1]$  and  $\xi$  is a random variable. Note that  $u(\cdot, 0)$  was assumed to be non-random. This was a novel setting comparing with the periodic conditions for the distributions, or with conditions from Klünger (2001) and Feng and Zhao (2012), or with conditions for expectations from Dokuchaev (2008).

The present paper addresses these and related problems again. We consider linear Dirichlet condition at the boundary of the state domain; the equations are of a parabolic type and are not necessary self-adjoint. The standard boundary value Cauchy condition at the one fixed time is replaced by a condition that mixes in one equation the terminal value of the solution and a functional of the entire solution. This setting covers conditions such as  $\theta^{-1} \int_0^\theta u(\cdot, t) dt = u(\cdot, T)$  a.s., as well as more general conditions. This is more general case comparing with the case considered in Dokuchaev (2012), where the boundary condition was connecting  $u(\cdot, T)$  with the expectations of the past values of  $u$ .

We present sufficient conditions for existence and regularity of the solutions in  $L_2$ -setting (Theorem 3.1). These results open a way to extend applications of backward SPDEs on the problems with non-local in time space boundary conditions.

## 2 The problem setting and definitions

We are given a standard complete probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and a right-continuous filtration  $\mathcal{F}_t$  of complete  $\sigma$ -algebras of events,  $t \geq 0$ . We assume that  $\mathcal{F}_0$  is the  $\mathbf{P}$ -augmentation of the set  $\{\emptyset, \Omega\}$ . We are given also a  $N$ -dimensional Wiener process  $w(t)$  with independent components; it is a Wiener process with respect to  $\mathcal{F}_t$ .

Assume that we are given a bounded open domain  $D \subset \mathbf{R}^n$  with  $C^2$ -smooth boundary  $\partial D$ . Let  $T > 0$  be given, and let  $Q \triangleq D \times [0, T]$ .

We will study the following boundary value problem in  $Q$

$$d_t u + (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N B_i \chi_i dt = \sum_{i=1}^N \chi_i(t) dw_i(t), \quad t \geq 0, \quad (2.1)$$

$$u(x, t, \omega)|_{x \in \partial D} = 0 \quad (2.2)$$

$$u(\cdot, T) - \Gamma u(\cdot) = \xi. \quad (2.3)$$

Here  $u = u(x, t, \omega)$ ,  $\varphi = \varphi(x, t, \omega)$ ,  $\xi = \xi(x, \omega)$ ,  $\chi_i = \chi_i(x, t, \omega)$ ,  $(x, t) \in Q$ ,  $\omega \in \Omega$ .

In (2.3),  $\Gamma$  is a linear operator that maps functions defined on  $Q \times \Omega$  to functions defined on  $D \times \Omega$ . For instance, the case where  $\Gamma u = u(\cdot, 0)$  is not excluded; this case corresponds to the periodic type boundary condition

$$u(\cdot, T) - u(\cdot, 0) = \xi. \quad (2.4)$$

In (2.1),

$$\mathcal{A}v = \sum_{i,j=1}^n b_{ij}(x, t, \omega) \frac{\partial^2 v}{\partial x_i \partial x_j}(x) + \sum_{i=1}^n f_i(x, t, \omega) \frac{\partial v}{\partial x_i}(x) + \lambda(x, t, \omega)v(x), \quad (2.5)$$

and

$$B_i v \triangleq \frac{dv}{dx}(x) \beta_i(x, t, \omega), \quad i = 1, \dots, N. \quad (2.6)$$

We assume that the functions  $b(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^{n \times n}$ ,  $\beta_j(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^n$ ,  $f(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}^n$ ,  $\lambda(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$ ,  $\chi_i(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$ , and  $\varphi(x, t, \omega) : \mathbf{R}^n \times [0, T] \times \Omega \rightarrow \mathbf{R}$  are progressively measurable with respect to  $\mathcal{F}_t$  for all  $x \in \mathbf{R}^n$ , and the function  $\xi(x, \omega) : \mathbf{R}^n \times \Omega \rightarrow \mathbf{R}$  is  $\mathcal{F}_0$ -measurable for all  $x \in \mathbf{R}^n$ .

In fact, we will also consider  $\varphi$  from wider classes. In particular, we will consider generalized functions  $\varphi$ .

We assume  $\lambda(x, t, \omega) \leq 0$  a.e., and  $b_{ij}, f_i, \chi_i$  are the components of  $b, f$ , and  $x$  respectively.

## Spaces and classes of functions

We denote by  $\|\cdot\|_X$  the norm in a linear normed space  $X$ , and  $(\cdot, \cdot)_X$  denote the scalar product in a Hilbert space  $X$ .

We introduce some spaces of real valued functions.

Let  $G \subset \mathbf{R}^k$  be an open domain, then  $W_q^m(G)$  denote the Sobolev space of functions that belong to  $L_q(G)$  together with the distributional derivatives up to the  $m$ th order,  $q \geq 1$ .

We denote by  $|\cdot|$  the Euclidean norm in  $\mathbf{R}^k$ , and  $\bar{G}$  denote the closure of a region  $G \subset \mathbf{R}^k$ .

Let  $H^0 \triangleq L_2(D)$ , and let  $H^1 \triangleq \overset{0}{W}_2^1(D)$  be the closure in the  $W_2^1(D)$ -norm of the set of all smooth functions  $u : D \rightarrow \mathbf{R}$  such that  $u|_{\partial D} \equiv 0$ . Let  $H^2 = W_2^2(D) \cap H^1$  be the space equipped with the norm of  $W_2^2(D)$ . The spaces  $H^k$  and  $W_2^k(D)$  are called Sobolev spaces, they are Hilbert spaces, and  $H^k$  is a closed subspace of  $W_2^k(D)$ ,  $k = 1, 2$ .

Let  $H^{-1}$  be the dual space to  $H^1$ , with the norm  $\|\cdot\|_{H^{-1}}$  such that if  $u \in H^0$  then  $\|u\|_{H^{-1}}$  is the supremum of  $(u, v)_{H^0}$  over all  $v \in H^1$  such that  $\|v\|_{H^1} \leq 1$ .  $H^{-1}$  is a Hilbert space.

We shall write  $(u, v)_{H^0}$  for  $u \in H^{-1}$  and  $v \in H^1$ , meaning the obvious extension of the bilinear form from  $u \in H^0$  and  $v \in H^1$ .

We denote by  $\bar{\ell}_k$  the Lebesgue measure in  $\mathbf{R}^k$ , and we denote by  $\bar{\mathcal{B}}_k$  the  $\sigma$ -algebra of Lebesgue sets in  $\mathbf{R}^k$ .

We denote by  $\bar{\mathcal{P}}$  the completion (with respect to the measure  $\bar{\ell}_1 \times \mathbf{P}$ ) of the  $\sigma$ -algebra of subsets of  $[0, T] \times \Omega$ , generated by functions that are progressively measurable with respect to  $\mathcal{F}_t$ .

We introduce the spaces

$$\begin{aligned} X^k(s, t) &\triangleq L^2([s, t] \times \Omega, \bar{\mathcal{P}}, \bar{\ell}_1 \times \mathbf{P}; H^k), \\ Z_t^k &\triangleq L^2(\Omega, \mathcal{F}_t, \mathbf{P}; H^k), \\ \mathcal{C}^k(s, t) &\triangleq C([s, t]; Z_T^k), \quad k = -1, 0, 1, 2, \\ \mathcal{X}_c^k &= L^2([0, T] \times \Omega, \bar{\mathcal{P}}, \bar{\ell}_1 \times \mathbf{P}; C^k(\bar{D})), \quad k \geq 0. \end{aligned}$$

The spaces  $X^k(s, t)$  and  $Z_t^k$  are Hilbert spaces.

We introduce the spaces

$$Y^k(s, t) \triangleq X^k(s, t) \cap \mathcal{C}^{k-1}(s, t), \quad k = 1, 2,$$

with the norm  $\|u\|_{Y^k(s, T)} \triangleq \|u\|_{X^k(s, T)} + \|u\|_{\mathcal{C}^{k-1}(s, T)}$ . For brevity, we shall use the notations  $X^k \triangleq X^k(0, T)$ ,  $\mathcal{C}^k \triangleq \mathcal{C}^k(0, T)$ , and  $Y^k \triangleq Y^k(0, T)$ .

We also introduce spaces  $\mathcal{C}_{PC}^k$  consisting of  $u \in \mathcal{C}^k$  such that either  $u \in \mathcal{C}^k$  or there exists  $\theta = \theta(u) \in [0, T]$  such that  $\|u(\cdot, t)\|_{Z_T^k}$  is bounded,  $u(\cdot, t)$  is continuous in  $Z_T^k$  in  $t \in [0, \theta]$ , and  $u(\cdot, t)$  is continuous in  $Z_T^k$  in  $t \in [\theta + \varepsilon, T]$  for any  $\varepsilon > 0$ .

Finally, we introduce the spaces

$$\begin{aligned}\mathcal{W} &\triangleq L^\infty([0, T] \times \Omega, \bar{\mathcal{P}}, \bar{\ell}_1 \times \mathbf{P}; L_\infty(D)) \cap \mathcal{C}_{PC}^0(0, T), \\ \mathcal{V} &\triangleq L^\infty(\Omega, \mathcal{F}_T, \mathbf{P}; L_\infty(D)).\end{aligned}$$

### Conditions for the coefficients

To proceed further, we assume that Conditions 2.1-2.3 remain in force throughout this paper.

**Condition 2.1** *The matrix  $b = b^\top$  is symmetric and bounded. In addition, there exists a constant  $\delta > 0$  such that*

$$y^\top b(x, t, \omega) y - \frac{1}{2} \sum_{i=1}^N |y^\top \beta_i(x, t, \omega)|^2 \geq \delta |y|^2 \quad \forall y \in \mathbf{R}^n, (x, t) \in D \times [0, T], \omega \in \Omega. \quad (2.7)$$

**Condition 2.2** *The functions  $f(x, t, \omega)$ ,  $\lambda(x, t, \omega)$ , and  $\beta_i(x, t, \omega)$  are bounded. These functions are differentiable in  $x$  for a.e.  $t, \omega$ , and the corresponding derivatives are bounded. In addition,  $b \in \mathcal{X}_c^3$ ,  $\hat{f} \in \mathcal{X}_c^2$ ,  $\lambda \in \mathcal{X}_c^1$ ,  $\beta_i \in \mathcal{X}_c^3$ , and  $\beta_i(x, t, \omega) = 0$  for  $x \in \partial D$ ,  $i = 1, \dots, N$ .*

Let  $\mathbb{I}$  denote the indicator function

**Condition 2.3** *The mapping  $\Gamma : \mathcal{W} \rightarrow \mathcal{V}$  is linear and continuous and such that  $\|\Gamma u\|_{\mathcal{V}} \leq \|u\|_{\mathcal{W}}$  for any  $u \in \mathcal{W}$ , and that there exists  $\theta < T$  such that  $\Gamma u = \Gamma(\mathbb{I}_{\{t \leq \theta\}} u)$ .*

**Example 2.1** Condition 2.3 is satisfied for the following operators:

- (i)  $\Gamma u = \kappa u(\cdot, 0)$ ,  $\kappa \in [-1, 1]$ ;
- (ii)  $(\Gamma u)(x, \omega) = \kappa u(x, t_1, \omega)$ ,  $t_1 \in [0, T]$ ;
- (iii)  $(\Gamma u)(x, \omega) = \zeta(\omega) u(x, t_1, \omega)$ ,  $t_1 \in [0, T]$ ,  $\zeta \in L_\infty(\Omega, \mathbf{P}, \mathcal{F}_T, \mathbf{P})$ ,  $|\zeta(\omega)| \leq 1$  a.s.;
- (iv)  $(\Gamma u)(x, \omega) = \alpha_1 u(x, t_1, \omega) + \alpha_2 u(x, t_2, \omega)$ ,  $t_1, t_2 \in [0, T]$ ,  $|\alpha_1| + |\alpha_2| \leq 1$ ;
- (v)

$$(\Gamma u)(x, \omega) = \int_0^\theta k(t) u(x, t, \omega) dt, \quad \theta \in [0, T], \quad k(\cdot) \in L_\infty(0, \theta), \quad \int_0^\theta |k(t)| dt \leq 1;$$

(vi)

$$(\Gamma u)(x, \omega) = \int_0^\theta dt \int_D k(t, y, x, \omega) u(y, t, \omega) dy,$$

where  $\theta \in [0, T)$ ,  $k(\cdot) : [0, \theta] \times D \times D \times \Omega$  is a bounded measurable function from  $L^\infty(\Omega, \mathcal{F}_T, \mathbf{P}, L^\infty([0, \theta] \times D \times D))$  such that

$$\operatorname{ess\,sup}_{(x, \omega) \in D \times \Omega} \int_0^\theta dt \int_D |k(t, x, y, \omega)| dy \leq 1.$$

Convex combinations of operators from this list are also covered.

Sometimes we shall omit  $\omega$ .

### The definition of solution

**Proposition 2.1** *Let  $\zeta \in X^0$ , let a sequence  $\{\zeta_k\}_{k=1}^{+\infty} \subset L^\infty([0, T] \times \Omega, \ell_1 \times \mathbf{P}; C(D))$  be such that all  $\zeta_k(\cdot, t, \omega)$  are progressively measurable with respect to  $\mathcal{F}_t$ , and let  $\|\zeta - \zeta_k\|_{X^0} \rightarrow 0$ . Let  $t \in [0, T]$  and  $j \in \{1, \dots, N\}$  be given. Then the sequence of the integrals  $\int_0^t \zeta_k(x, s, \omega) dw_j(s)$  converges in  $Z_t^0$  as  $k \rightarrow \infty$ , and its limit depends on  $\zeta$ , but does not depend on  $\{\zeta_k\}$ .*

*Proof* follows from completeness of  $X^0$  and from the equality

$$\mathbf{E} \int_0^t \|\zeta_k(\cdot, s, \omega) - \zeta_m(\cdot, s, \omega)\|_{H^0}^2 ds = \int_D dx \mathbf{E} \left( \int_0^t (\zeta_k(x, s, \omega) - \zeta_m(x, s, \omega)) dw_j(s) \right)^2.$$

**Definition 2.1** Let  $\zeta \in X^0$ ,  $t \in [0, T]$ ,  $j \in \{1, \dots, N\}$ , then we define  $\int_0^t \zeta(x, s, \omega) dw_j(s)$  as the limit in  $Z_t^0$  as  $k \rightarrow \infty$  of a sequence  $\int_0^t \zeta_k(x, s, \omega) dw_j(s)$ , where the sequence  $\{\zeta_k\}$  is such as in Proposition 2.1.

**Definition 2.2** Let  $u \in Y^1$ ,  $\chi_i \in X^0$ ,  $i = 1, \dots, N$ , and  $\varphi \in X^{-1}$ . We say that equations (2.1)-(2.2) are satisfied if

$$\begin{aligned} u(\cdot, t, \omega) &= u(\cdot, T, \omega) + \int_t^T (\mathcal{A}u(\cdot, s, \omega) + \varphi(\cdot, s, \omega)) ds \\ &\quad + \sum_{i=1}^N \int_t^T B_i \chi_i(\cdot, s, \omega) ds - \sum_{i=1}^N \int_t^T \chi_i(\cdot, s) dw_i(s) \end{aligned}$$

for all  $r, t$  such that  $0 \leq r < t \leq T$ , and this equality is satisfied as an equality in  $Z_T^{-1}$ .

Note that the condition on  $\partial D$  is satisfied in the sense that  $u(\cdot, t, \omega) \in H^1$  for a.e.  $t, \omega$ . Further,  $u \in Y^1$ , and the value of  $u(\cdot, t, \omega)$  is uniquely defined in  $Z_T^0$  given  $t$ , by the definitions of the corresponding spaces. The integrals with  $dw_i$  in (2.8) are defined as elements of  $Z_T^0$ . The integral with  $ds$  in (2.8) is defined as an element of  $Z_T^{-1}$ . In fact, Definition 2.2 requires for (2.1) that this integral must be equal to an element of  $Z_T^0$  in the sense of equality in  $Z_T^{-1}$ .

### 3 The main results

**Theorem 3.1** *Problem (2.1)-(2.3) has a unique solution  $(u, \chi_1, \dots, \chi_N)$  in the class  $Y^1 \times (X^0)^N$  for any  $\varphi \in \mathcal{W}$  and  $\xi \in Z_T^0$ . This solution is such that  $u \in \mathcal{W}$ . In addition,*

$$\|u\|_{\mathcal{W}} + \|u\|_{Y^1} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C (\|\varphi\|_{\mathcal{W}} + \|\xi\|_{\mathcal{V}}), \quad (3.1)$$

where  $C > 0$  does not depend on  $\varphi$  and  $\xi$ .

### 4 Proofs

Let  $s \in (0, T]$ ,  $\varphi \in X^{-1}$  and  $\Phi \in Z_s^0$ . Consider the problem

$$\begin{aligned} d_t u + (\mathcal{A}u + \varphi) dt + \sum_{i=1}^N B_i \chi_i(t) dt &= \sum_{i=1}^N \chi_i(t) dw_i(t), \quad t \leq s, \\ u(x, t, \omega)|_{x \in \partial D}, \\ u(x, s, \omega) &= \Phi(x, \omega). \end{aligned} \quad (4.1)$$

The following lemma represents an analog of the so-called "the first energy inequality", or "the first fundamental inequality" known for deterministic parabolic equations (see, e.g., inequality (3.14) from Ladyzhenskaya (1985), Chapter III).

**Lemma 4.1** *Assume that Conditions 2.1–2.3 are satisfied. Then problem (4.1) has an unique solution a unique solution  $(u, \chi_1, \dots, \chi_N)$  in the class  $Y^1 \times (X^0)^N$  for any  $\varphi \in X^{-1}(0, s)$ ,  $\Phi \in Z_s^0$ , and*

$$\|u\|_{Y^1(0, s)} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C (\|\varphi\|_{X^{-1}(0, s)} + \|\Phi\|_{Z_s^0}), \quad (4.2)$$

where  $C > 0$  does not depend on  $\varphi$  and  $\xi$ .

(See, e.g., Dokuchaev (1991) or Theorem 4.2 from Dokuchaev (2010)).

Note that the solution  $u = u(\cdot, t)$  is continuous in  $t$  in  $L_2(\Omega, \mathcal{F}, \mathbf{P}, H^0)$ , since  $Y^1(0, s) = X^1(0, s) \cap \mathcal{C}^0(0, s)$ .

Introduce operators  $L_s : X^{-1}(0, s) \rightarrow Y^1(0, s)$  and  $\mathcal{L}_s : Z_s^0 \rightarrow Y^1(0, s)$ , such that  $u = L_s\varphi + \mathcal{L}_s\Phi$ , where  $(u, \chi_1, \dots, \chi_N)$  is the solution of problem (4.1) in the class  $Y^2 \times (X^1)^N$ . By Lemma 4.1, these linear operators are continuous.

Introduce operators  $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$  and  $\mathcal{T} : X^{-1} \rightarrow Z_T^0$  such that  $\mathcal{Q}\Phi = \Gamma\mathcal{L}_T\Phi$  and  $\mathcal{T}\varphi = \Gamma L_T\varphi$ , i.e.  $\mathcal{Q}\Phi + \mathcal{T}\varphi = \Gamma u$ , where  $u$  is the solution in  $Y^1$  of problem (4.1) with  $s = T$ ,  $\varphi \in X^{-1}$ , and  $\Phi \in Z_T^0$ .

It is easy to see that if the operator  $\Gamma : Y^1 \rightarrow Z_T^0$  is continuous, then the operators  $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$  and  $\mathcal{T} : X^{-1} \rightarrow Z_T^0$  are linear and continuous. In particular,  $\|\mathcal{Q}\| \leq \|\Gamma\|\|\mathcal{L}_T\|$ , where  $\|\mathcal{Q}\|$ ,  $\|\Gamma\|$ , and  $\|\mathcal{L}_T\|$ , are the norms of the operators  $\mathcal{Q} : Z_T^0 \rightarrow Z_T^0$ ,  $\Gamma : Y^1 \rightarrow Z_T^0$ , and  $\mathcal{L}_T : Z_T^0 \rightarrow Y^1$ , respectively.

**Lemma 4.2** *Assume that the operator  $\Gamma : Y^1 \rightarrow Z_T^0$  is continuous. If the operator  $(I - \mathcal{Q})^{-1} : Z_T^0 \rightarrow Z_T^0$  is also continuous then problem (4.1) has a unique solution  $(u, \chi_1, \dots, \chi_N)$  in the class  $Y^1 \times (X^0)^N$  for any  $\varphi \in X^{-1}$ ,  $\Phi \in Z_T^0$ . For this solution,*

$$u = L_T\varphi + \mathcal{L}_T(I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi) \quad (4.3)$$

and

$$\|u\|_{Y^1(0,s)} + \sum_{i=1}^N \|\chi_i\|_{X^0} \leq C (\|\varphi\|_{X^{-1}(0,s)} + \|\Phi\|_{Z_s^0}),$$

where  $C = C(\mathcal{P})$  does not depend on  $\varphi$  and  $\xi$ .

*Proof of Lemma 4.2.* For brevity, we denote  $u(\cdot, t) = u(x, t, \omega)$ . Clearly,  $u \in Y^1$  is the solution of problem (2.1)-(2.3) with some  $(\chi_1, \dots, \chi_N) \in (X^0)^N$  if and only if

$$u = \mathcal{L}_T u(\cdot, T) + L_T\varphi, \quad (4.4)$$

$$u(\cdot, T) - \Gamma u = \xi. \quad (4.5)$$

Since  $\Gamma u = \mathcal{Q}u(\cdot, T) + \mathcal{T}\varphi$ , equation (4.5) can be rewritten as

$$u(\cdot, T) - \mathcal{Q}u(\cdot, T) - \mathcal{T}\varphi = \xi. \quad (4.6)$$

By the continuity of  $(I - \mathcal{Q})^{-1}$ , equation (4.6) can be rewritten as

$$u(\cdot, T) = (I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi).$$

Therefore, equations (4.4)-(4.5) imply that

$$u = L_T\varphi + \mathcal{L}_T u(\cdot, T) = L_T\varphi + \mathcal{L}_T(I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi).$$



Further, let us show that if (4.3) holds then equations (4.4)-(4.5) hold. Let  $u$  be defined by (4.3). Since  $u = L_T\varphi + \mathcal{L}_T u(\cdot, T)$ , it follows that  $u(\cdot, T) = (I - \mathcal{Q})^{-1}(\xi + \mathcal{T}\varphi)$ . Hence

$$u(\cdot, T) - \mathcal{Q}u(\cdot, T) = \xi + \mathcal{T}\varphi,$$

i.e.,  $u(\cdot, T) - \Gamma \mathcal{L}_T u(\cdot, T) = \xi + \mathcal{T}\varphi = \xi + \Gamma L_T \varphi$ . Hence

$$u(\cdot, T) - \Gamma[\mathcal{L}_T u(\cdot, T) + L_T \varphi] = \xi.$$

This means that (4.4)-(4.5) hold. Then the proof of Lemma 4.2 follows.  $\square$

Let functions  $\tilde{\beta}_i : \mathcal{Q} \times \Omega \rightarrow \mathbf{R}^n$ ,  $i = 1, \dots, M$ , be such that

$$2b(x, t, \omega) = \sum_{i=1}^N \beta_i(x, t, \omega) \beta_i(x, t, \omega)^\top + \sum_{j=1}^M \tilde{\beta}_j(x, t, \omega) \tilde{\beta}_j(x, t, \omega)^\top,$$

and  $\tilde{\beta}_i$  has the similar properties as  $\beta_i$ . (Note that, by Condition 2.1,  $2b > \sum_{i=1}^N \beta_i \beta_i^\top$ ).

Let  $\tilde{w}(t) = (\tilde{w}_1(t), \dots, \tilde{w}_M(t))$  be a new Wiener process independent on  $w(t)$ . Let  $a \in L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbf{R}^n)$  be a vector such that  $a \in D$ . We assume also that  $a$  is independent from  $(w(t) - w(t_1), \hat{w}(t) - \hat{w}(t_1))$  for all  $t > t_1 > s$ . Let  $s \in [0, T)$  be given. Consider the following Ito equation

$$\begin{aligned} dy(t) &= f(y(t), t) dt + \sum_{i=1}^N \beta_i(y(t), t) dw_i(t) + \sum_{j=1}^M \tilde{\beta}_j(y(t), t) d\tilde{w}_j(t), \\ y(s) &= x. \end{aligned} \tag{4.7}$$

Let  $y(t) = y^{a,s}(t)$  be the solution of (4.7), and let  $\tau^{a,s} \triangleq \inf\{t \geq s : y^{a,s}(t) \notin D\}$ .

**Lemma 4.3** *For any  $\vartheta > 0$ , there exists  $\nu = \nu(\vartheta) \in (0, 1)$  that depends only on  $D, \mathcal{A}, B_j$  and such that  $\mathbf{P}_s(\tau^{x,s} > s + \vartheta) \leq \nu$  a.s. for all  $s \geq 0$ , and for any  $x \in D$ .*

Note that if the functions  $f(x, t, \omega) = f(x)$  and  $\beta(x, t, \omega) = \beta(x)$  are non-random and constant in  $t$ , then existence of  $\nu \in (0, 1)$  such that  $\mathbf{P}(\tau^{a,s} > s + \vartheta) \leq \nu$  ( $\forall a, s$ ) is obvious.

*Proof of Lemma 4.3.* In this proof, we will follow the approach from Dokuchaev (2004), p.296. Let  $\mu = (\hat{f}, \beta, x, s)$ .

Clearly, there exists a finite interval  $D_1 \triangleq (d_1, d_2) \subset \mathbf{R}$  and a bounded domain  $D_{n-1} \subset \mathbf{R}^{n-1}$  such that  $D \subset D_1 \times D_{n-1}$ .

For  $(x, s) \in D \times [0, T)$ , let  $\tau_1^{x,s} \triangleq \inf\{t \geq s : y_1^{x,s}(t) \notin D_1\}$ , where  $y_1^{x,s}(t)$  is the first component of the vector  $y^{x,s}(t) = (y_1^{x,s}(t), \dots, y_n^{x,s}(t))$ . We have that

$$\mathbf{P}_s(\tau^{x,s} > s + \vartheta) \leq \mathbf{P}_s(\tau_1^{x,s} > s + \vartheta) = \mathbf{P}_s(y_1^{x,s}(t) \in D_1 \forall t \in [s, s + \vartheta]). \tag{4.8}$$

Let

$$M^\mu(t) \triangleq \sum_{k=1}^N \int_s^t h_k(y^{x,s}(r), r) dw_i(r) + \sum_{k=N+1}^{N+M} \int_s^t h_k(y^{x,s}(r), r) d\tilde{w}_i(r), \quad t \geq s,$$

where  $h = (h_1, \dots, h_{N+M})$  is a vector that represents the first row of the matrix

$$(\beta_1, \dots, \beta_N, \hat{\beta}_1, \dots, \hat{\beta}_M)$$

with the values in  $\mathbf{R}^{n \times (N+M)}$ .

Let  $\hat{D}_1 \triangleq (d_1 + K_1, d_2 + K_2)$ , where  $K_1 \triangleq -d_2 - \vartheta \sup_{x,t,\omega} |\hat{f}_1(x, t, \omega)|$ ,  $K_2 \triangleq -d_1 + \vartheta \sup_{x,t} |\hat{f}_1(x, t, \omega)|$ . Clearly,  $\hat{D}_1$  depends only on  $n, D$ , and  $c_f$ . It is easy to see that

$$\mathbf{P}_s(y_1^{x,s}(t) \in D_1 \quad \forall t \in [s, s + \vartheta]) \leq \mathbf{P}_s(M^\mu(t) \in \hat{D}_1 \quad \forall t \in [s, s + \vartheta]). \quad (4.9)$$

Further,

$$h(y^{x,s}(t), t)^\top h(y^{x,s}(t), t) = |h(y^{x,s}(t), t)|^2 \in [\delta, c_\beta], \quad (4.10)$$

where

$$\delta = \inf_{x,s,\omega, \xi \in \mathbf{R}^n: |\xi|=1} 2\xi^\top b(x, t, \omega)\xi, \quad c_\beta = \sup_{x,s,\omega, \xi \in \mathbf{R}^n: |\xi|=1} 2\xi^\top b(x, t, \omega)\xi.$$

Clearly,  $M^\mu(t)$  is a martingale vanishing at  $s$  conditionally given  $\mathcal{F}_s$  with quadratic variation process

$$[M^\mu]_t \triangleq \int_s^t |h(y^{x,s}(r), r)|^2 dr, \quad t \geq s.$$

Let  $\theta^\mu(t) \triangleq \inf\{r \geq s : [M^\mu]_r > t - s\}$ . Note that  $\theta^\mu(s) = s$ , and the function  $\theta^\mu(t)$  is strictly increasing in  $t > s$  given  $(x, s)$ . By Dambis–Dubins–Schwarz Theorem (see, e.g., Revuz and Yor (1999)), the process  $B^\mu(t) \triangleq M(\theta^\mu(t))$  is a Brownian motion conditionally given  $\mathcal{F}_s$  vanishing at  $s$ , i.e.,  $B^\mu(s) = 0$ , and  $M^\mu(t) = B^\mu(s + [M^\mu]_t)$ . Clearly,

$$\begin{aligned} \mathbf{P}_s(M^\mu(t) \in \hat{D}_1 \quad \forall t \in [s, s + \vartheta]) &= \mathbf{P}_s(B^\mu(s + [M^\mu]_t) \in \hat{D}_1 \quad \forall t \in [s, s + \vartheta]) \\ &\leq \mathbf{P}_s(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + [M^\mu]_{s+\vartheta}]). \end{aligned} \quad (4.11)$$

By (4.10),  $[M^\mu]_{s+\vartheta} \geq \delta\vartheta$  a.s. for all  $x, s$ . Hence

$$\mathbf{P}_s(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + [M^\mu]_{s+\vartheta}]) \leq \mathbf{P}_s(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + \delta\vartheta]). \quad (4.12)$$

By (4.8)–(4.9) and (4.11)–(4.12), it follows that

$$\sup_\mu \mathbf{P}_s(\tau^{x,s} > s + \vartheta) \leq \nu \triangleq \sup_\mu \mathbf{P}_s(B^\mu(r) \in \hat{D}_1 \quad \forall r \in [s, s + \delta\vartheta]),$$

and  $\nu = \nu(\mathcal{P}) \in (0, 1)$ . This completes the proof of Lemma 4.3.  $\square$

*Proof of Theorem 3.1.* For  $t \geq s$ , set

$$\gamma^{a,s}(t) \triangleq \exp \left( - \int_s^t \lambda(y^{a,s}(t), t) dt \right).$$

Let  $\Phi \in \mathcal{V}$  and  $\varphi \in \mathcal{W}$  be bounded. By Theorem 4.1 from Dokuchaev (2011) again, we have that, for any  $s \in [0, T]$  and  $u = L_T \xi + \mathcal{L}_T \Phi$ ,  $u(\cdot, s)$  can be represented as

$$u(x, s, \omega) = \mathbf{E} \left\{ \gamma^{x,s}(T) \Phi(y^{x,s}(T)) \mathbb{I}_{\{\tau^{x,s} \geq T\}} + \int_s^{\tau^{x,s}} \gamma^{x,s}(t) \varphi(y^{x,s}(t), t, \omega) dt \mid \mathcal{F}_s \right\}. \quad (4.13)$$

This equality holds in  $Z_s^0$  and for a.e.  $x, \omega$ . It follows that

$$\sup_{s \in [0, T]} \|u(\cdot, s)\|_{\mathcal{V}} \leq \|\Phi\|_{\mathcal{V}} + T \|\varphi\|_{\mathcal{W}}. \quad (4.14)$$

Hence

$$\|\mathcal{L}_T \Phi\|_{\mathcal{W}} \leq \|\Phi\|_{\mathcal{V}}, \quad \|L_T \varphi\|_{\mathcal{W}} \leq T \|\varphi\|_{\mathcal{W}}. \quad (4.15)$$

By the assumptions on  $\Gamma$ , it follows that  $\|\Gamma u\|_{\mathcal{V}} \leq \|u\|_{\mathcal{W}}$ . It follows that the operators  $\mathcal{Q} = \Gamma \mathcal{L}_T : \mathcal{V} \rightarrow \mathcal{V}$  and  $\mathcal{T} : \mathcal{W} \rightarrow \mathcal{V}$  are bounded. Let  $\|\mathcal{Q}\|_{\mathcal{V}, \mathcal{V}}$  be the norm of the operator  $\mathcal{Q} : \mathcal{V} \rightarrow \mathcal{V}$ .

It follows from (4.14) and from the properties of  $\Gamma$  that  $\|\mathcal{Q}\|_{\mathcal{V}, \mathcal{V}} \leq 1$ . Let us refine this estimate.

Let  $u = \mathcal{L}_T \Phi$ ,  $s \in [0, T]$ . Let  $y(t) = y^{x,s}(t)$  be the solution of Ito equation (4.7) with the initial condition  $y(s) = x$ . For the brevity, we will use notations  $\mathbf{P}_s(\cdot) \triangleq \mathbf{P}(\cdot | \mathcal{F}_s)$  and  $\mathbf{E}_s(\cdot) \triangleq \mathbf{E}(\cdot | \mathcal{F}_s)$ . By (4.13), it follows that

$$\begin{aligned} \|u(\cdot, s)\|_{\mathcal{V}} &= \text{ess sup}_{x, \omega} \mathbf{E}_s \gamma^{x,s}(T) \Phi(y^{x,s}(T)) \mathbb{I}_{\{\tau^{x,s} \geq T\}} \\ &\leq \text{ess sup}_{x, \omega} \left[ \mathbf{E}_s \mathbb{I}_{\{\tau^{x,s} \geq T\}}^2 \right]^{1/2} \text{ess sup}_{x, \omega} \left[ \mathbf{E}_s \Phi(y^{x,s}(T))^2 \right]^{1/2} \\ &\leq \text{ess sup}_{x, \omega} \left[ \mathbf{E}_s \mathbb{I}_{\{\tau^{x,s} \geq T\}}^2 \right]^{1/2} \|\Phi\|_{\mathcal{V}} = \text{ess sup}_{x, \omega} \mathbf{P}_s(\tau^{x,s} \geq T)^{1/2} \|\Phi\|_{\mathcal{V}}. \end{aligned}$$

If  $s < \theta$  then  $\{\tau^{x,s} \geq T\} \subseteq \{\tau^{x,s} \geq s + \vartheta\}$ , where  $\vartheta \triangleq T - \theta > 0$ . Hence

$$\|u(\cdot, s)\|_{\mathcal{V}} \leq \text{ess sup}_{x, \omega} \mathbf{P}_s(\tau^{x,s} \geq s + \vartheta)^{1/2} \|\Phi\|_{\mathcal{V}}, \quad s \leq \theta.$$

By Lemma 4.3, it follows that there exists  $\nu = \nu(\vartheta, \mathcal{P}) \in (0, 1)$  such that  $\mathbf{P}_s(\tau^{x,s} \geq s + \vartheta) < \nu$  a.s. It follows that

$$\|u(\cdot, s)\|_{\mathcal{V}} \leq \nu^{1/2} \|\Phi\|_{\mathcal{V}}, \quad s \leq \theta$$

and

$$\|\mathbb{I}_{\{s \leq \theta\}} u\|_{\mathcal{W}} \leq \nu^{1/2} \|\Phi\|_{\mathcal{V}}.$$

By the assumptions on  $\Gamma$ , it follows that

$$\|\Gamma u\|_{\mathcal{V}} = \|\Gamma(\mathbb{I}_{\{s \leq \theta\}} u)\|_{\mathcal{V}} \leq \nu^{1/2} \|\Phi\|_{\mathcal{V}}, \quad s \leq \theta.$$

It follows that  $\|\mathcal{Q}\|_{\mathcal{V}, \mathcal{V}} \leq \nu^{1/2} < 1$ . Hence the operator  $(I - \mathcal{Q})^{-1} : \mathcal{V} \rightarrow \mathcal{V}$  is bounded. Let

$$u = L_T \varphi + \mathcal{L}_T (I - \mathcal{Q})^{-1} (\xi + \mathcal{T} \varphi). \quad (4.16)$$

By the assumptions on  $\Gamma$  and by (4.13)-(4.15), it follows that  $\xi + \mathcal{T} \varphi = \xi + \Gamma L_T \varphi \in \mathcal{V} \subset Z_T^0$ . Hence  $(I - \mathcal{Q})^{-1} (\xi + \mathcal{T} \varphi) \in \mathcal{V} \subset Z_T^0$ . By the properties of  $\mathcal{L}_T$  and  $L_T$ , it follows that  $u \in Y^1$ . By (4.13)-(4.15) again, it follows that  $u \in \mathcal{W}$ . Similarly to the proof of Lemma 4.2, it can be shown that  $u$  is a part of the unique solution  $(u, \chi_1, \dots, \chi_N) \in Y^1 \times (X^0)^N$  of problem (2.1)-(2.3). Estimate (3.1) follows from the continuity of the corresponding operators in (4.16). Then the proof of Theorem 3.1 follows.  $\square$

## Acknowledgment

This work was supported by ARC grant of Australia DP120100928 to the author.

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